JOURNAL OF FUNCTIONAL ANALYSIS 79, 32–43 (1988)

Fundamental Approximate Identities and Quasi-multipliers of Simple AF C^*-Algebras

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Communicated by the Editors

Received August 28, 1986; revised April 24, 1987

We study C^*-algebras with fundamental approximate identities as a generalization of stable C^*-algebras, and show that the only separable, simple AF C^*-algebras for which the quasi-multipliers equal the left plus right multipliers are unital or elementary.

1. Introduction

Let A be a C^*-algebra and A^{**} its enveloping Von Neumann algebra. An element $x \in A^{**}$ is a multiplier of A if xa, $ax \in A$ for all $a \in A$, x is a right multiplier if $ax \in A$ for all $a \in A$, and x is a quasi-multiplier if $axb \in A$ for all $a, b \in A$. We denote the multipliers, left multipliers, right multipliers, and quasi-multipliers by $M(A)$, $LM(A)$, $RM(A)$, and $QM(A)$, respectively.

The question was raised by Akemann and Pedersen [1] whether $QM(A) = LM(A) + RM(A)$. This problem has been studied in [6, 15, 16]. For example, it has been shown in [15] that a separable stable C^*-algebra A satisfies $QM(A) = LM(A) + RM(A)$ if and only if A is scattered and $\lambda(\hat{A}) < \infty$ (we will define this later). In [6], L. G. Brown showed the connection between this problem and the problem of perturbations of C^*-algebras. In this paper, fundamental approximate identity has been studied, and, using that, we give some generalization of the result mentioned above. Moreover, we answer the quasi-multiplier question for separable simple AF C^*-algebras. We also give applications for perturbation problems.

2. Fundamental Approximate Identities

Let A be a C^*-algebra and p, q projections in A. We say $p \precsim q$ if there is $u \in A$ such that $uu^* = p$ and $u^*u \leqslant q$.

0022-1236/88 $3.00

DEFINITION 1. Let A be a C^*-algebra. An approximate identity $\{e_n\}$ consisting of projections is fundamental if $0 \leqslant e_{n+1} - e_n \lesssim e_n - e_{n-1}$ $(e_0 = 0)$.

If K denotes the C^*-algebra consisting of all compact operators on the separable Hilbert space, then K has a fundamental approximate identity. More generally, if A is a unital C^*-algebra, then $A \otimes K$ has a fundamental approximate identity. It is easy to see that $A \otimes K$ contains a C^*-subalgebra B such that the hereditary C^*-subalgebra generated by B is A, and there is a *-hmomorphism from B onto K.

LEMMA 1. *Let A be a C^*-algebra with a fundamental approximate identity. Then there is a C^*-subalgebra B of A such that the hereditary C^*-subalgebra generated by B is A, and there is a *-homomorphism from B onto K.*

Proof. Let $\{e_n\}$ be a fundamental approximate identity for A, and put $f_n = e_n - e_{n-1}$ $(e_0 = 0)$.

We claim that there are $p_k^{(i)} \in A$, $i \leqslant k$, and $u_k^{(i)} \in A$, $i \leqslant k-1$, $i = 1, 2, \ldots$, $k = 1, 2, \ldots$, such that

(i) $p_k^{(i)}$ are projections;

(ii) $p_k^{(i)} \leqslant p_l^{(i)} \leqslant f_i$, if $l \leqslant k$ and $p_k^{(k)} = f_i$;

(iii) $(u_k^{(i)})(u_k^{(i)})^* = p_k^{(i)}$, $(u_k^{(i)})^* (u_k^{(i)}) = f_k = p_k^{(k)}$.

We shall prove the claim by induction on k. Assume that the claim is true for all $k' \leqslant k$. Since $p_k^{(i)} \sim p_k^{(k)} = f_k \gtrsim f_{k+1}$, there are $u_{k+1}^{(i)} \in A$ such that $(u_{k1}^{(i)})(u_{k+1}^{(i)})^* \leqslant p_k^{(i)}$, $(u_{k+1}^{(i)})(u_{k+1}^{(i)})^*$ is a projection and $(u_{k+1}^{(i)})^* (u_{k+1}^{(i)}) = f_{k+1} = p_{k+1}^{(k+1)}$. Clearly $p_{k+1}^{(i)}$, $u_k^{(i)}$ satisfy (i), (ii), and (iii).

It is easy to see that the C^*-subalgebra A_k generated by $u_k^{(i)}$, $i \leqslant k-1$, and $p_k^{(i)}$, $i \leqslant k$, is isomorphic to M_k (the $k \times k$ matrix algebra).

Let $e'_k = \sum_{i=1}^{k} p_k^{(i)}$ and q be a weak limit point (in A^{**}) of $\{e'_k\}$.

For fixed n, $e_n e'_k$ is decreasing and $e_n e'_k = \sum_{i=1}^{n} p_k^{(i)}$, for $k \geqslant n$, and is a decreasing sequence of projections. So $e_n e'_k$ converges strongly to $e_n q$. Hence $e_n q \in (A_+)_m$. Thus $e_n q$ is an upper semi-continuous function on the quasi-state space of A (see [17, 3.11]). By a standard compactness argument, $e_n q \neq 0$ and $e_n q$ is a projection in A^{**}. Now $e_n q \uparrow q$, so q is a nonzero projection in A^{**}. Furthermore, $e'_k \to q$, strongly.

Since e'_k commutes with every element in A_i, $i \leqslant k$, we conclude that q commutes with every element in A_i. Let A_0 be the C^*-algebra generated by $\{A_i, i = 1, 2, \ldots\}$. Then q commutes with A_0, and thus there is a *-homomorphism from A_0 onto qA_0. For fixed i, the same argument used above shows that $p_k^{(i)} \downarrow p_i^{(i)} q \neq 0$. It is easy to check that qA_i is isomorphic to M_i, and thus qA_0 is isomorphic to K.

Since A_0 contains $\{e_k\}$, the hereditary C^*-subalgebra generated by A_0 is A.

LEMMA 2. *Let A be an antiliminal C^*-algebra with a fundamental approximate identity. Then there is a C^*-subalgebra B of A such that the hereditary C^*-subalgebra generated by B is A and there is a $*$-homomorphism from B onto $C(X)\otimes K$ with X a compact Hausdorff space with a nonempty perfect subset.*

Proof. We keep the notations as in the proof of Lemma 1. Let π be the homomorphism $A_0 \to qA_0 \cong K$. Let $\tilde{\pi}$ be an extension of π to A. Clearly $\tilde{\pi}(\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}) \supset \pi(\bigcap_{k=1}^{\infty} p_k^{(1)}A_0 p_k^{(1)})$. Since $qp_1^{(1)} \in \pi(\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)})$ and $qp_1^{(1)} \neq 0$, we conclude that $\tilde{\pi}(\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}) \neq 0$. Thus $\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}$ is a nonzero hereditary C^*-subalgebra of A. Since A is an antiliminal, $\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}$ is not type I. In particular, $\bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}$ is not scattered. Hence there is a $a \in \bigcap_{k=1}^{\infty} p_k^{(1)}Ap_k^{(1)}$ such that $0 \leqslant a \leqslant 1$ and $\sigma(a)$ has a nonempty perfect subspace (see [11]). Clearly $a \leqslant p_k^{(1)}$ for all k. Since $p_k^{(1)} \downarrow qp_1^{(1)}$, $a \leqslant qp_1^{(1)}$. Let $p_i = p_i^{(i)}q$ (p_i a projection in A^{**}). Let B_1 be the C^*-subalgebra generated by p_1 and a. Thus $B_1 \cong C(X)$ with X a compact Hausdorff space with a nonempty perfect subset. Let $u_i = u_{i+1}^{(i)}(q)$, $i = 1, 2, \ldots$, and B_n be the C^*-subalgebra generated by p_i, a, and u_i, $i = 1, 2, \ldots, n$. It is a routine exercise to check that B_n can be represented by $n \times n$ matrices with entries in $C(X)$, or $B_n \cong C(X)\otimes M_n$. It is then easy to see that the C^*-subalgebra generated by $\{B_n,\ n = 1, 2, \ldots\}$ is isomorphic to $C(X)\otimes K$. Let A' be the C^*-subalgebra generated by A_0 and a. Then q commutes with each element in A'. Moreover, $qA' \cong C(X)\otimes K$. Since A' contains $\{e_n\}$, the hereditary C^*-subalgebra generated by A' is A.

There exists a C^*-algebra A without fundamental approximate identity which has a C^*-subalgebra B such that the hereditary C^*-subalgebra generated by B is A, and there is a $*$-homomorphism from B onto K.

EXAMPLES. Let $A_1 = C_0 \otimes M_n$. Then it is easy to check that A_1 has no fundamental approximate identity. Let $A = A_1 \oplus K$. Then A has no fundamental approximate identity, and there is a $*$-homomorphism from A onto K.

We have the following extension theorem.

THEOREM 1. *Let*

$$0 \to K \to A \to B \to 0$$

be a short exact sequence of C^*-algebras. Suppose that B has a fundamental approximate identity. Then A has a fundamental approximate identity.

Proof. We may assume that K is an ideal of A, $B=A/K$, and π is the canonical homomorphism from A to A/K. Let $\{\bar{e}_n\}$ be a fundamental approximate identity for A/K and $f_n=\bar{e}_n-\bar{e}_{n-1}$ $(e_0=0)$, $n=1, 2, \ldots$. It follows from the projection-lifting theorem ([9, Lemma 9.7] or [5]) that there is a projection $p_1 \in A$ such that $\pi(p_1)=f_1$. Suppose we have $p_1, p_2, \ldots, p_m$ in A such that $\pi(p_i)=f_i$, $p_i \gtrsim p_{i+1}$, and p_i are mutually orthogonal. Let $v_m \in B$ such that $v_m^* v_m$ is a projection in B, $v_m^* v_m \leqslant f_m$, and $v_m v_m^* = f_{m+1}$. Let $u \in A$ such that $\pi(u)=v_m$. Replacing u by $(1-\sum_{i=1}^m p_i)\, u p_m$, we may assume that $u^*u \in p_m A p_m$ and $uu^* \in (1-\sum_{i=1}^m p_i)\ A(1-\sum_{i=1}^m p_i)$. Since $\pi(p_m-u^*u)=0$, $p_m-u^*u \in p_m K p_m$. There is a projection $q \in p_m K p_m$ with

$$\|(p_m-u^*u)(p_m-q)\|<1.$$

Let $x=(p_m-q)\, u^*u(p_m-q)$, then

$$\|x-(p_m-q)\| = \|(p_m-q)(p_m-u^*u)(p_m-q)\|<1,$$

since $p_m-q \leqslant p_m$. It follows that x is invertible in $(p_m-q)\, A(p_m-q)$. Thus we have a polar decomposition for $u(p_m-q)$. We take $h=x^{1/2}$ and $u_m=u(p_m-q)\, x^{-1/2}$ (the inverse is taken in $(p_m-q)\, A(p_m-q)$), then $u(p_m-q)=u_m h$. So u_m is a partial isometry. Moreover,

$$\pi(u_m)=\pi(u)\, \pi(p_m-q)\, \pi(x^{-1/2})=v_m,$$

since $q \in \operatorname{Ker} \pi$ and $\pi(x)=v_m^* v_m=f_m$. So $u_m u_m^*$ is a projection and $\pi(u_m u_m^*)=f_{m+1}$.

Let $p_{m+1}=u_m u_m^*$. Since $p_{m+1} \in (1-\sum_{i=1}^m p_i)\, A(1-\sum_{i=1}^m p_i)$, it follows that $p_{m+1} p_i = p_i p_{m+1}=0$, if $i \leqslant m$. Since $u_m^* u_m$ is a projection in $(p_m-q)\, A(p_m-q)$, $u_m^* u_m \leqslant p_m$, i.e., $p_{m+1} \lesssim p_m$. Thus we have a sequence of projections $\{p_m\}$ in A such that $\pi(p_m)=f_m$, $p_i p_j=0$ if $i \neq j$ and $p_{i+1} \lesssim p_i$. Let $p=\sum_{i=1}^\infty p_i$. Then p is an open projection in A^{**} and $\{\sum_{i=1}^m p_i\}$ is an approximate identity for the hereditary C^*-subalgebra $pA^{**}p \cap A$ of A. Let π^{**} be the extension of π to A^{**}. Since $\pi^{**}(p)$ is the identity of B^{**}, we conclude that $(1-p) \in K^{**}=M(K)$. So $(1-p)$ $K(1-p) \subset K$. We may assume that $(1-p)\, K(1-p) \cong K$. Hence $(1-p)\, K(1-p)$ has a fundamental approximate identity $\{\varepsilon_m\}$. Let $d_m=\varepsilon_m-\varepsilon_{m-1}$ $(\varepsilon_0=0)$, $q_m=p_m+d_m$, and $e_m=\sum_{i=1}^m q_i$. It is easy to check that $q_{m+1} \lesssim q_m$ and e_m, q_m are projections. We claim that $\{e_m\}$ is an approximate identity for $A_0=(pA^{**}p \cap A+K)^-$.

Let $y=x+z$, where $x \in pA^{**}p \cap A$, $z \in K$. We need only prove that $\|y(1-e_m)\| \to 0$. Clearly, $\|x(1-e_m)\| \to 0$. Since $pz^*zp \in pA^{**}p \cap A$,

$$\begin{aligned}\|zp(1-e_m)\|^2 &= \|(1-e_m)\, pz^*zp(1-e_m)\| \\ &\leqslant \|pz^*zp(1-e_m)\| \to 0.\end{aligned}$$

Moreover,

$$\begin{aligned}\|z(1-p)(1-e_m)\|^2 &= \|(1-e_m)(1-p)\,z^*z(1-p)(1-e_m)\| \\ &\leqslant \|(1-p)\,z^*z(1-p)(1-\varepsilon_m)\| \to 0.\end{aligned}$$

So $\{e_m\}$ is an approximate identity for A_0. Let $a=\sum_{m=1}^{\infty}(1/2^m)\,e_m$, then a is a strictly positive element of A_0. Let f be a positive linear functional on A. If $f(K)=0$, f can be viewed as a positive linear functional on A/K. Since $\pi(a)$ is also a strictly positive element of A/K, we have $f(a)>0$. If $f(K)\neq 0$, then clearly $f(a) = f(\sum_{m=1}^{\infty}(1/2^m)(\sum_{i=1}^{m} p_i)) + f(\sum_{m=1}^{\infty}(1/2^m)\,\varepsilon_m) > 0$. Thus, A_0 has an element a which is strictly positive in A. We conclude that the hereditary C^*-subalgebra generated by A_0 is A. Finally, we conclude that $\{e_n\}$ is a fundamental approximate identity for A.

Let A be a separable simple $AF\,C^*$-algebra and G its simple dimension group. Fix $u\in G^+\setminus\{0\}$. Let $S_u(G)$ be the collection of positive (i.e., $\tau(G^+)\geqslant 0$) homomorphisms $\tau\colon G\to \mathbf{R}$ with $\tau(u)=1$. Then $S_u(G)$ is a convex compact subset of the locally convex space $\mathbf{R}^G$ of all functions $f\colon G\to\mathbf{R}$ with the product topology. Moreover, $S_u(G)$ is a Choquet simplex (see [9, Chap. 4]), i.e., each point of $S_u(G)$ is the barycenter of a unique probability measure on the set of extreme points $E(S_u(G))$. A Choquet simplex is called Bauer simplex if the set of extreme points is closed. Let $g(\tau)=\sup\{\tau([e_n]), n=1,2,\ldots\}$, where $\{e_n\}$ is an approximate identity of A consisting of projections. We say A has a continuous scale if $g(\tau)$ is a continuous function on $S_u(G)$ for some (hence for all) $u\in G^+\setminus\{0\}$.

THEOREM 2. *Let A be a nonunital, separable simple $AF\,C^*$-algebra. Then A has a fundamental approximate identity, if one of the following conditions holds*:

(1) *A is stable*;

(2) *A has a continuous scale*;

(3) *$S_u(G)$ is a Bauer simplex for some $u\in G^+\setminus\{0\}$.*

Proof. (1) A is stable. Then A has no finite trace. Let p be a nonzero projection of A. By the proof of Theorem 4.10 in [4], A is isomorphic to $pAp\otimes K$. Since pAp is unital, A has a fundamental approximate identity.

(3) $S_u(G)$ is a Bauer simplex. Let $\Gamma(G)$ be the scale of A. Let $\theta\colon G\to \mathrm{Aff}(S_u(G))$ be its affine representation (see [9, Chap. 4]), and let $H=\theta(G)$. It follows from [9, Corollary 4.2] that $G^+=\{a\in G\colon \theta(a)\gg 0\}\cup\{0\}$ and $\Gamma(G)\cap\operatorname{Ker}\theta=\{0\}$. So θ maps $\Gamma(G)$ bijectively to a scale of H and preserves the order.

Suppose that A is a C^*-algebraic inductive limit of A_n, where each A_n is a finite-dimensional C^*-algebra. If e_n is the identity of A_n then $\{e_n\}$ forms an approximate identity. Let $g = \sup\{\theta([e_n]) : r = 1, 2, \ldots\}$, so that g is an affine function (the range may include infinity) on the Bauer simplex $S_u(G)$. We claim that $\theta(\Gamma(G)) = \{f \in H : 0 < f < g \text{ or } f = 0\}$. For each projection $p \in A$, there is an n such that $\|a - p\| < \frac{1}{2}$ for some $a \in (A_n)_{\text{s.a.}}$. It follows from [9, Lemmas A.81 and A.82] that $p \sim q$ for some projection $q \in A_n$. Thus $e_n \gtrsim p$. Hence $\theta([p]) < g$. If $f \in H$, and $0 < f < g$, then for each $t \in S_u(G)$, there is an integer n such that

$$\theta([e_n])(t) > f(t).$$

Since both $\theta(\{e_n])$ and f are continuous, there is a neighborhood $U(t)$ of t such that

$$\theta([e_n])(t) > f(t), \qquad \text{for all } t \in U(t).$$

By the compactness of $S_u(G)$, we may assume that

$$\theta([e_n])(t) > f(t), \qquad \text{for all } t \in S_u(G).$$

By [9, p. 43], $f \in \theta(\Gamma(G))$.

Let $E[S_u(G)]$ be the (closed) set of extreme points of $S_u(G)$. It is easy to find a sequence of continuous functions $\{g_n\}$ on $E[S_u(G)]$ that converges to g on $E[S_u(G)]$ satisfying (i) $g_n > 0$, (ii) $g_n \uparrow g$, (iii) $g_{n+1} - g_n < g_n - g_{n-1}$. It follows from [2, Theorem II.4.3] or [3] that we can extend the g_n's affinely and uniquely to $S_u(G)$. We still use the notations g_n for the extensions. Since H is dense in $\text{Aff}(S_u(G))$ (see [9, Theorem 4.4]), we may assume that $g_n \in H$. Thus $g_n \in \theta(\Gamma(G))$.

Let q_k be projections in A such that $\theta([q_k]) = g_k$. Let $m(k) = \min\{n : [q_k] \leqslant [e_n]\}$, and $N(k) = \min\{n : [e_{m(k)}] \leqslant [q_n]\}$. By a compactness argument, we can show easily that both $m(k)$ and $N(k)$ go to infinity as n goes to infinity. We now construct a sequence of projections $\{p_k\} \subset A$. Take $p_{N(1)} \in A_{m(N(1))}$ such that $p_{N(1)} \geqslant e_{m(1)}$ and $[p_{N(1)}] = [q_{N(1)}]$. We may assume that $q_i \leqslant p_{N(1)}$, $i = 1, 2, \ldots, N(1)$. Since $q_i \lesssim q_{i+1}$, we can find $p_i \leqslant p_{N(1)}$ such that $p_{i-1} \leqslant p_i$, $[p_i] = [q_i]$, $i = 1, 2, \ldots, N(1)$ $(p_0 = 0)$. Take $p_{N^*(2)} \in A_{m^*(2)}$, where $N^*(2) = \max\{N(m(N(1))), N(2)\}$, $m^*(2) = \max\{m(N(m(N(1)))), m(N(2))\}$, such that with $j = \max(m(N(1)), m(2))$, $p_{N^*(2)} \geqslant e_j$ $(\gtrsim q_i \gtrsim p_{N(1)}$, $i = N(1)+1, N(1)+2, \ldots, N^*(2)-1)$, and $[p_{N^*(2)}] = [q_{N^*(2)}]$. So we can find $p_i \in A_j$ such that $p_{N(1)} \leqslant p_{i-1} \leqslant p_i$ $(\leqslant p_{N^*(2)})$, $i = N(1)+1, N(1)+2, \ldots, N^*(2)-1$. Continuing by induction, we obtain a sequence of projections $\{p_k\}$ in A such that $p_k \leqslant p_{k+1}$, $[p_k] = [q_k]$, and $p_{N^*(k)} \geqslant e_{m(k)}$. So $\{p_k\}$ forms an approximate identity. Since $[p_{k+1} - p_k] = [q_{k+1}] - [q_k]$, by the construction of g_n, $[p_{k+1} - p_k] \leqslant [p_k - p_{k-1}]$. So $p_{k+1} - p_k \lesssim p_k - p_{k-1}$. We conclude that $\{p_k\}$ is a fundamental approximte identity.

(2) A has a continuous scale. Let g be as in (3). By definition, g is continuous, and it is clear that $\theta([e_n])\uparrow g$. By Dini's theorem, $g-\theta([e_n])\downarrow 0$ uniformly on $S_u(G)$. We define $\{n_k\}$ recursively as follows: if n_k has been defined, we can find n_{k+1} such that

$$\sup_{t\in S_u(G)}\{g(\tau)-\theta([e_{n_{k+1}}])(\tau)\}\leqslant\tfrac{1}{2}\inf_{\tau\in S_u(G)}\{g(\tau)-\theta([e_{n_k}])(\tau)\}.$$

Let $f_k=e_{n_k}-e_{n_{k-1}}$ $(e_0=0)$. Then $[f_k]>[f_{k+1}]$. Hence $f_k\gtrsim f_{k+1}$. We see that $\{e_{n_k}\}$ forms a fundamental approximate identity for A.

Let A be a nonelementary simple $AF\,C^*$-algebra described in Theorem 2. Since A is antiliminal, we may assume that $A\subset B(H)$, the bounded linear operators on a separable Hilbert space, and $A\cap K=\{0\}$. Let π be the homomorphism $B(H)\to B(H)/K$. $\pi(A)\cong A$. Let $B=\pi^{-1}(A)$. By Theorem 1, B has a fundamental approximate identity.

3. Quasi-multipliers

Recall that a C^*-algebra is scattered if it is type I and has scattered spectrum $\hat{A}$ (see [11]). Let X be a scattered topological space. We define $X_{[0]}=X$, $X_{[1]}=X\setminus\{\text{isolated points of } X\}$. If $X_{[\alpha]}$ is defined for some ordinal number α, define $X_{[\alpha+1]}=X_\alpha\setminus\{\text{isolated points of } X_\alpha\}$; if β is a limit ordinal, define $X_{[\beta]}=\bigcap_{\alpha<\beta}X_{[\alpha]}$. We define $\lambda(X)=\alpha$, if α is the least ordinal such that $X_{[\alpha]}$ is discrete. Since X is scattered, $\lambda(X)$ is well defined.

The following is a generalization of [15, Theorem 6.3].

THEOREM 3. *Let A be a C^*-algebra with a fundamental approximate identity and B a unital C^*-algebra. Then $QM(B\otimes A)=LM(B\otimes A)+RM(B\otimes A)$ implies that B is scattered and $\lambda(\hat{B})<\infty$.*

Proof. It follows from Lemma 1 that there is a C^*-subalgebra A_0 of A such that the hereditary C^*-subalgebra generated by A_0 is A and there is a $*$-homomorphism from A_0 onto K. Thus the hereditary C^*-subalgebra generated by $B\otimes A_0$ is $B\otimes A$ and there is a homomorphism φ such that $\varphi(B\otimes A_0)=B\otimes K$. By Theorem 3.1 of [15], if $QM(B\otimes A)=LM(B\otimes A)+RM(B\otimes A)$, then $QM(B\otimes A_0)=LM(B\otimes A_0)+RM(B\otimes A_0)$. Since $\varphi(B\otimes A_0)=B\otimes K$, it follows from Theorem 6.3 of [15] that B is scattered and $\lambda(\hat{B})<\infty$.

THEOREM 4. *Let A be an antiliminal C^*-algebra with a fundamental approximate identity. Then $QM(A)\neq LM(A)+RM(A)$.*

Proof. By Lemma 2, there is a C^*-algebra B such that the hereditary C^*-subalgebra generated by B is A, and there is a $*$-homomorphism from

B onto $C(X)\otimes K$ with X being a compact Hausdorff space with a nonempty perfect subset. It follows from Theorem 3.1 of [15] that if $QM(A)=LM(A)+RM(A)$, then $QM(B)=LM(B)+RM(B)$. However, by Theorem 6.3 of [15], $QM(B)\neq LM(B)+RM(B)$.

By Theorems 2 and 4, the quasi-multipliers of nonelementary simple AF C^*-algebras described in Theorem 2 are not the left multipliers plus the right multipliers. However, we have the following more precise theorem.

THEOREM 5. *Let A be a separable simple AF C^*-algebra. Then $QM(A)=LM(A)+RM(A)$ if and only if A is unital or A is isomorphic to K.*

Proof. If A is unital or A is isomorphic to K, then $QM(A)=M(A)$.

Now suppose A is a nonelementary and nonunital simple AF C^*-algebra.

Let $\{e_n\}$ be an approximate identity for A consisting of projections. Let G be the simple dimensional group of A with scale $\Gamma(G)$. Fix $[e_1]=u\in G^+\setminus\{0\}$, and let $\theta\colon G\to \mathrm{Aff}[S_u(G)]$ be its affine representation (see [9, Chap. 4]). Let $g_n=\theta([e_n])\in\theta(\Gamma(G))$. Put $h_1=g_1$. There is $h_2\in\theta(\Gamma(G))$ such that $h_2>g_1$ and $0<g_2-h_2<h_1$. By induction we may construct a sequence $\{h_k\}\subset\theta(\Gamma(G))$ such that $h_{2k-1}=g_k$, $g_k<h_{2k}<g_{k+1}$, and $g_{k+1}-h_{2k}<\min_{i<2k}(h_i-h_{i-1})$. In fact, if $h_1, ..., h_{2k}$ have been constructed, let $h_{2k+1}=g_{2k+1}$. Find $h_{2(k+1)}\in\theta(\Gamma(G))$ such that $g_{k+1}<h_{2(k+1)}<g_{k+2}$ and $g_{k+2}-h_{2k+2}<\min_{i<2k+2}(h_i-h_{i-1})$. This is possible since every $h\in\theta(G)$ with $0<h<g_{k+2}$ belongs to $\theta(\Gamma(G))$, and $\theta(G)$ is dense in $\mathrm{Aff}[S_u(G)]$. So we have projections $\{e'_n\}\subset A$ such that $e'_{2k-1}=e_k$, $e_k<e'_{2k}<e_{k+1}$, and $e'_{2k+1}-e'_{2k}\lesssim e'_n-e'_{n-1}$ for all $n<2k$.

We claim that there are $p_k^{(i)}\in A$, $i\leqslant 2k+1$, $i=1, 2, ...,$ and $k=1, 2, ...,$ and $u_k^{(i)}\in A$, $i\leqslant 2k$, $i=1, 2, ...,$ and $k=1, 2, ...,$ such that

(i) $p_k^{(i)}$ are projections;

(ii) $p_k^{(i)}\leqslant p_l^{(i)}\leqslant e'_i-e'_{i-1}$, if $l\leqslant k$ and $p_k^{(2k+1)}=e'_{2k+1}-e'_{2k}$;

(iii) $(u_k^{(i)})(u_k^{(i)})^*=p_k^{(i)}$, $(u_k^{(i)})^*(u_k^{(i)})=p_k^{(2k+1)}$.

We shall prove the claim by induction on k. Suppose that the claim is true for $k'\leqslant k$. Since $p_k^{(i)}\sim p_k^{(2k+1)}=e'_{2k+1}-e'_{2k}\gtrsim e'_{2k+3}-e'_{2k+2}$, there are $u_{k+1}^{(i)}\in A$ such that $(u_{k+1}^{(i)})^*(u_{k+1}^{(i)})=p_{k+1}^{(2k+3)}=e'_{2k+3}-e'_{2k+2}$, $(u_{k+1}^{(i)})(u_{k+1}^{(i)})^*=p_{k+1}^{(i)}\leqslant p_k^{(i)}$, where $p_{k+1}^{(i)}$ are projections. Clearly, $u_{k+1}^{(i)}, p_{k+1}^{(i)}$ satisfy (i), (ii), and (iii).

Let $\varepsilon_k=\sum_{i=1}^{2k+1}p_k^{(i)}$ and q be a weak limit point of $\{\varepsilon_k\}$. For fixed n, $e'_{2n+1}\varepsilon_k$ is decreasing and $e'_{2n+1}\varepsilon_k=\sum_{i=1}^{m}p_k^{(i)}$, where $m=\min(2k+1, 2n+1)$ are projections. As in the proof of Lemma 1, e_{2n+1}, q and q are nonzero projections in A^{**}, and$\varepsilon_k\to q$, strongly.

Let A_k be the C^*-subalgebra generated by $p_k^{(i)}$ and $u_k^{(i)}$, $i \leqslant 2k+1$. Then A_k is isomorphic to M_{2k+1}. Let A_0 be the C^*-subalgebra generated by $\{A_k, k=1, 2, ...\}$. As in the proof of Lemma 1, q commutes with each element in A_0 and A_0 is isomorphic to $qA_0 \cong K$.

As in the proof of Lemma 2, there is a in $\bigcap_{k=1}^{\infty} p_k^{(1)} A P_k^{(1)}$ such that $0 \leqslant a \leqslant qp_1^{(1)}$ and $\sigma(a)$ has a nonempty perfect subset. If B is the C^*-subalgebra generated by a and A_0, then by the proof of Lemma 2, there is a *-homomorphism from B onto $C(X) \otimes K$. It follows from [15, Theorem 6.3] that $QM(B) \neq LM(B) + RM(B)$. If $\varepsilon'_m = \sum_{i=1}^{m} (p_i^{(2i+1)})$, then $\{e'_m\}$ forms an approximate identity for B and $\varepsilon'_m \varepsilon'_n = \varepsilon'_n \varepsilon'_m = \varepsilon'_n$, if $m > n$. There is $x \in QM(A)_{\text{s.a.}}$ such that for every m_k

$$\sup \left\{ \left\| \sum_{k=1}^{N} (1 - \varepsilon'_{m_k}) x (\varepsilon'_{m_k} - \varepsilon'_{m_{k-1}}) \right\| \right\} = \infty$$

(by [15, Theorem 2.3]). For every m,

$$\begin{aligned} e'_{2m+1} x e'_{2m+1} &= e'_{2m+1} \sum_{i=1}^{\infty} (p_i^{(2i)} + p_i^{(2i+1)}) \, x \sum_{i=1}^{\infty} (p_i^{(2i)} + p_i^{(2i+1)}) \, e'_{2m+1} \\ &= e'_{2m+1} \sum_{i=1}^{m} (p_i^{(2i)} + p_i^{(2i+1)}) \, x \sum_{i=1}^{m} (p_i^{(2i)} + p_i^{(2i+1)}) \, e'_{2m+1} \\ &= \varepsilon'_m x \varepsilon'_m \in B \subset A. \end{aligned}$$

Since $x \in B^{**} \subset A^{**}$, we conclude that $x \in QM(A)_{\text{s.a.}}$. For every m_k,

$$\begin{aligned} &\left\| \sum_{k=1}^{N} (1 - e'_{2(m_k)+1}) \, x (e'_{2(m_k)+1} - e'_{2(m_{k-1})+1}) \right\| \\ &\quad = \left\| \sum_{k=1}^{N} (1 - \varepsilon'_{m_k}) \, x (\varepsilon'_{m_k} - \varepsilon_{m_{k-1}}) \right\|. \end{aligned}$$

Since $\{e'_{2m+1}\}$ is an approximate identity satisfying $e'_{2m+1} e'_{2n+1} = e'_{2n+1} e'_{2m+1} = e'_{2n+1}$ if $m > n$, by [15, Theorem 2.3], $x \notin LM(A) + RM(A)$.

COROLLARY 1. *Let A be a nonelmentary separable matroid C^*-algebra without identity. Then*

$$QM(A) \neq LM(A) + RM(A).$$

COROLLARY 2. *For each C^*-algebra A which is not of type I, there is a C^*-subalgebra B of A such that $QM(B) \neq LM(B) + RM(B)$.*

Proof. By [17, 6.7.4], there is a C^*-subalgebra B_1 of A such that there is a *-homomorphism from B_1 onto a nonelementary separable matroid

C^*-algebra M_∞ with identity. Let $\{p_n\}$ be an increasing sequence of projections of M_∞ such that p_n are strictly smaller than the identity. Let $M'_\infty = (\bigcup p_n M_\infty p_n)^-$. Then M'_∞ is a nonunital, nonelementary, and separable matroid C^*-algebra. Let φ be the homomorphism of B_1 onto M_∞, $B = \varphi^{-1}(M'_\infty)$. By Corollary 1, $QM(B) \neq LM(B) + RM(B)$.

However, as in [15, Example 8.3], we know that there is an antiliminal C^*-algebra A such that $M(A) \neq QM(A)$ but $QM(A) = LM(A) + RM(A)$.

Question. Is it true that the only simple C^*-algebras A with $QM(A) = LM(A) + RM(A)$ are unital or K?

We have the following weaker result.

THEOREM 6. *Suppose that A is a σ-unital simple C^*-algebra. Then $QM(A) = M(A)$ if and only if A is unital or A is isomorphic to K.*

Proof. We assume that A is nonunital and not isomorphic to K. Let $\{e_n\}$ be an approximate identity satisfying $e_n e_m = e_m e_n = e_m$ if $n > m$. Let $f_n = e_n - e_{n-1}$ $(e_0 = 0)$, then $f_n f_{n+j+1} = f_{n+j+1} f_n = 0$, if $j = 1, 2, \ldots$. If $e_k A e_k$ are finite dimensional for all k, then A is a separable AF algebra. By Theorem 4, $QM(A) \neq M(A)$. Thus we may assume that $e_1 A e_1$ is infinite dimensional. Let $\{p_i\}$ be a sequence in $e_1 A e_1$ such that $p_i \in A_+$ and $p_i p_j = 0$ if $i \neq j$.

Let $J = \{a \mid f_k x a = 0 \text{ for all } x \in A\}$. Then J is a closed ideal and $J \neq A$ $(f_k \notin J)$. Since A is simple, $J = \{0\}$. Thus for every k, there is $x_k \in A$ such that $y_k = f_k x_k p_k \neq 0$ and $\|y_k\| = 1$. Define $y = \sum_{k=1}^\infty y_k$. It can be shown as in [15, Lemma 2.2] that $\|y\| \leqslant 2$ and $y \in A^{**}$. For every e_n, $e_n y = e_n \sum_{k=1}^{n+1} y_k \in A$. Hence $y \in RM(A)$. However, $y \notin M(A)$. In fact, $y e_2 = \sum_{k=1}^\infty y_k e_2 = \sum_{k=1}^\infty y_k = y$. So we need only show that $y \notin A$. For every k, $(1 - e_k)\, y = \sum_{i=k+1}^\infty y_i + (1 - e_k)\, y_k$. Let π be a faithful representation of A^{**} on H_π and $v_i \in q_i H_\pi$, where q_i is the range projection of p_i. Then $\|(1 - e_k)\, y v_i\| = \|y_i v_i\|$, if $i > k$. We may take v_i such that $\|y_i v_i\| > \frac{1}{2}$. Thus $\|(1 - e_k)\, y\| > \frac{1}{2}$ for all k, which implies that $y \notin A$.

4. APPENDIX

In [5], L. G. Brown showed the connection between the problem of whether $QM(A) = LM(A) + RM(A)$ and the problem of perturbations of C^*-algebras. Perturbations of C^*-algebras have been considered in several different ways (see [7, 8, 12–14]). One of them is to ask whether an almost isometric ($\|\varphi\| - 1$ and $\|\varphi^{-1}\| - 1$ are small) complete order automorphism of a C^*-algebra is close to an isometry.

THEOREM 7 (Brown [7]). *If A is a σ-unital simple C^*-algebra and $QM(A) \neq LM(A) + RM(A)$, then there exists a sequence $\{\varphi_n\}$ of complete order automorphisms of A such that* $\lim_{n \to \infty} \|\varphi_n\| = 1$ *and*

$$\lim_{n \to \infty} \|\varphi_n^{-1}\| = 1$$

but $\inf\{\|\theta - \varphi_n\|, n = 1, 2, ..., \theta \text{ automorphisms of } A\} > 0$.

COROLLARY 3. *If A is a separable, nonelementary simple AF C^*-algebra without identity, then there exists a sequence $\{\varphi_n\}$ of complete order automorphisms of A such that*

$$\lim_{n \to \infty} \|\varphi_n\| = 1 \quad \text{and} \quad \lim_{n \to \infty} \|\varphi_n^{-1}\| = 1$$

but

$$\inf\{\|\theta - \varphi_n\|, n = 1, 2, ..., \theta \text{ automorphisms of } A\} > 0.$$

Proof. $QM(A) \neq LM(A) + RM(A)$, so Theorem 7 applies.

ACKNOWLEDGMENTS

The author wishes to express his sincere thanks to Professor Lawrence G. Brown for his valuable suggestions. The author would also like to thank the referee for his or her help.

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